The Method of brackets

with applications to computing Feynman Diagrams

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1 Introduction

In this short note we will give a short introduction to the method of brackets following the articles [ABFP23], [Amd12], [GS07], [GS07] and [GMS10]. A proof of Ramanujan Master theorem was first published in the book "Ramanujan: Twelve Lectures on subjects Suggested by His Life and Work" by G.H Hardy. Where Hardy proved the following theorem,

Theorem 1 *Let us write $s = x + iy$ and let ϕ be a holomorphic and $|\phi(s)| \leq Ce^{C_1 x + C_2|y|}$, throughout $H(\delta) = \{z \in \mathbb{C} \mid x \geq -\delta\}$ for some constant C and $\delta > 0$. Whenever ϕ satisfies these conditions we say that $\phi \in R(C_1, C_2, \delta)$. Further suppose $0 < x < \delta$ and let $\Phi(s)$ be the holomorphic function $\phi(0) - s\phi(1) + s^2\phi(2)\ldots$ which converges throughout $H(\delta)$ because of our assumptions then,*

$$\int_0^\infty \Phi(t)t^{s-1}\,dt = \phi(-s)\cdot\frac{\pi}{\sin(\pi s)}.$$

A simple corollary of this theorem proves that

$$\int_0^\infty \frac{t^{s-1}}{1+t}\,dt = \frac{\pi}{\sin(\pi s)},$$

for $0 < Re(s) < 1$. Which can also be computed using the Feynman trick. Furthermore a easy exponential version of this theorem can be deduced by rewritting $\phi(s)$ as $\frac{\lambda(s)}{\Gamma(1+s)}$ and using the formula $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$.

Corollary 1 *Let $\lambda \in R(C_1, C_2, \delta)$. Further suppose $0 < x < \delta$ and let $\Lambda(s)$ be the holomorphic function $\lambda(0) - \frac{s}{1!}\lambda(1) + \frac{s^2}{2!}\lambda(2)\ldots$ which converges throughout $H(\delta)$ because of our assumptions then,*

$$\int_0^\infty \Lambda(t)t^{s-1}\,dt = \lambda(-s)\Gamma(s).$$

1.1 The Method of Brackets

The method of brackets gives a straightfiward generalization of the master theorem which allows for computation of multivariate integrals. The method of brackets is a formal technique to computing certain integrals. We with some notation, we let $\langle s\rangle$ denote the informal integral $\int_0^\infty x^{s-1}\,dx$ and let ϕ_n be shorthand for $\frac{(-1)^n}{\Gamma(1+n)} = \frac{(-1)^n}{\Gamma(n+1)}$ which we call the indicator of n. Now from expanding e^{-x} and interchanging summation and integration it follows trivially that

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}\,dx = \sum_{n=0}^\infty \phi_n\langle s+n\rangle.$$

Now we can begin with a formal derivation which will give us our first rule this method. Consider the formula

$$\frac{\Gamma(s)}{C^s} = \int_0^\infty t^{s-1}e^{-Ct}\,dt,$$

then by letting $C = (x_1 + \ldots + x_n)^{-1}$ we have

$$(x_1 + \ldots + x_n)^s = \frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1} e^{-(x_1+\ldots+x_n)t}\, dt =$$

$$\int_0^\infty t^{-s-1} e^{-x_1 t} \cdots e^{-x_n t}\, dt = \sum_{m_1,\ldots,m_n}^{\infty} \phi_{m_1} \cdots \phi_{m_n} x_1^{m_1} \cdots x_n^{m_n} \frac{\langle -s + m_1 + \ldots + m_n \rangle}{\Gamma(-s)}$$

From this derivation we assign the following three rules,

1. For the expression $(a_1 + \ldots + a_n)^\alpha$ we assign the expression

$$\sum_{m_1=1,\ldots,m_r=1}^{\infty} \phi_{m_1,\ldots,m_r} a_1^{m_1} a_2^{m_2} \cdots a_r^{m_r} \frac{\langle -s + m_1 + \ldots + m_n \rangle}{\Gamma(-s)}.$$

2. For the series of brackets

$$\sum_{n=1}^{\infty} \phi_n f(n) \langle an + b \rangle$$

we assign the value

$$\frac{1}{a} f(n^*) \Gamma(-n^*)$$

where n^* is the solution to $an + b = 0$.

3. For the double series

$$\sum_{m_1,m_2} \phi_{m_1} \phi_{m_2} f(m_1, m_2) \langle a_{1,1} m_1 + a_{1,2} m_2 + c_1 \rangle \langle a_{2,1} m_1 + a_{2,2} m_2 + c_2 \rangle$$

we assign the value

$$\frac{1}{|det(A)|} f(n_1^*, n_2^*) \Gamma(-n_1^*) \Gamma(-n_2^*)$$

where $A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$ and n_1^*, n_2^* are the solutions to $\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

Notice now that rule 2. with $a = 1, b = s$ gives us the master theorem so rule 3 can be seen as a multivariate generalization of the master theorem. We will provide a rigrous proof using distributions of the master theorem in section 5.

2 Applications of The Method

Consider the integral

$$\int_0^\infty e^{-x^2}\, dx = \sum_{n=0}^{\infty} \phi_n \langle 2n + 1 \rangle.$$

Then by rule 2. this is $\frac{1}{2}\Gamma(\frac{1}{2}) = \sqrt{\pi}/2$. Similarly it follows that

$$\int_0^\infty e^{-x^s} dx = \sum_{n=0}^{\infty} \phi_n \langle sn + 1 \rangle = \frac{1}{s}\Gamma(1/s).$$

Which can also easily be seen from the substitution $u = x^s$. However we can further generalize this formula using the method of brackets since

$$\int_0^\infty \int_0^\infty e^{-(x_1+x_2)^\alpha} x_1^{s_1-1} x_2^{s_2-1}\, dx_1 dx_2 = \int_0^\infty \int_0^\infty \sum_n \phi_n x_1^{s_1-1} x_2^{s_2-1} (x_1 + x_2)^{\alpha n}\, dx_1 dx_2,$$

and now from the first rule we get

$$\int_0^\infty \int_0^\infty \sum_n \phi_n x_1^{s_1-1} x_2^{s_2-1}(x_1+x_2)^{\alpha n}\ dx, = \sum_n \phi_n \int_0^\infty \int_0^\infty \sum_{j,k} \phi_j \phi_k x_1^{j+s_1-1} x_2^{k+s_2-1} \langle -\alpha n + k + j\rangle / \Gamma(-\alpha n) =$$

$$\sum_{n,j,k} \phi_n \phi_j \phi_k \langle j + s_1\rangle \langle k + s_2\rangle \langle -\alpha n + k + j\rangle / \Gamma(-\alpha n).$$

Now to find our formula using rule 3 requires us to solve the equations

$$\begin{bmatrix} -\alpha & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n \\ j \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ -s_1 \\ -s_2 \end{bmatrix}.$$

Which gives the solutions $j^* = -s_1, k^* = -s_2, n = -\frac{s_1+s_2}{\alpha}$ and the determinant of the matrix is $-\alpha$ hence,

$$\sum_{n,j,k} \phi_n \phi_j \phi_k \langle j + s_1\rangle \langle k + s_2\rangle \langle -\alpha n + k + j\rangle / \Gamma(-\alpha n) = \frac{1}{\alpha} \frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+s_2)} \Gamma\left(\frac{s_1+s_2}{\alpha}\right).$$

Which in turn gives us the computation

$$\int_0^\infty \int_0^\infty e^{-(x_1+x_2)^\alpha} x_1^{s_1-1} x_2^{s_2-1}\ dx_1 dx_2 = \frac{1}{\alpha} \frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+s_2)} \Gamma\left(\frac{s_1+s_2}{\alpha}\right).$$

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