A short remark on Wielandt's theorem

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Abstract

In 1939 Helmut Wielandt proved a function theoretic characterization for the Gamma function. In this short note we extend his theorem to the multivariate beta function and use it to prove a simple, well-known product formula. We will simply follow the same techniques used in the article [1].

1 Two definitions of the multivariate β-function

There are usually two ways one can define the multivariate beta function. The first and simplest way is as in [2] where they write,

$$B(z_1, \ldots, z_n) = \frac{\Gamma(z_1) \cdots \Gamma(z_n)}{\Gamma(z_1 + z_2 + \ldots + z_n)},$$

where $\Gamma(z_i)$ denotes the Γ-function. Another way is as discussed in [2] is as,

$$B(z_1, \ldots, z_n) = \int_{E_{n-1}} \prod_{i=1}^{n} t_i^{z_i} \, \mathrm{dt}_1 \ldots \mathrm{dt}_{n-1},$$

where $t_n = 1 - t_1 - t_2 \ldots t_n$ and

$E_{n-1} = \{(t_1, \ldots, t_{n-1}) \in [0,1]^{n-1} \mid \sum_{i=1}^{n} t_i \leq 1\}$. Because of a lack of reference we will give a short proof of this proposition below,

2 Wielandt's theorem

Wielandt's theorem says that given any analytic function f in one variable on the right half-plane $\{z \in \mathbb{C} \mid \text{Re } z > 0\}$ such that $f(z+1) = zf(z)$ and f is bounded on $\{z \in \mathbb{C} \mid 2 \geq \text{Re } z \geq 1\}$. Then $f(z) = c \cdot \Gamma(z)$ where c is a constant equal to $f(1)$.

Now a easy generalization of this theorem is the following,

Theorem 2.1. *Let $f(z_1, \ldots, z_n)$ be a analytic function on $\{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid$ Re $z_i > 0$ for $1 \leq i \leq n\}$ such that,*

- *For each i,*

$$f(z_1,\ldots,z_i+1,\ldots,z_n)=\frac{z_i}{z_1+\ldots+z_n}f(z_1,\ldots,z_n)$$

- *For each i and each fixed $c_1,\ldots,c_{i-1},c_{i+1},\ldots,c_n\in\mathbb{C}$ then f is bounded on the strip $\{c_1\}\times\cdots\times\{z_i\in\mathbb{C}\mid 2\geq Re\, z_i\geq 1\}\times\cdots\times c_n$*

then $f(z_1,\ldots,z_n)=cB(z_1,\ldots,z_n)$ where c is a constant.

Proof. The proof follows directly from Wielandt's theorem. Let $g(z_1,\ldots,z_n)=\Gamma(z_1+\ldots+z_n)f(z_1,\ldots,z_n)$ then for each i we have $g(z_1,\ldots,1+z_i,\ldots,z_n)=z_i\cdot g(z_1,\ldots,z_n)$ and so if we fix each variable besides z_i then by Wielandt's theorem we can write $g(z_1,\ldots,z_n)=k_i(z_1,\ldots,\hat{z}_i,\ldots,z_n)\cdot\Gamma(z_i)$. Now $k_i(z_1,\ldots,\hat{z}_i,\ldots,z_n)$ is analytic since $\frac{1}{\Gamma(z_i)}$ is analytic on our domain. Now it follows that $k_1\cdot\Gamma(z_1)=k_j\cdot\Gamma(z_j)$ and so

$$\frac{k_1}{\Gamma(z_j)}=\frac{k_j}{\Gamma(z_1)},$$

however we can notice now that the right side of the equation does not depend on j and so for each $j\neq i$ then $\frac{k_1}{\Gamma(z_j)}$ is constant with respect to j and hence,

$$\frac{k_1}{\Gamma(z_2)\cdots\Gamma(z_n)}$$

does not depend on z_j since

$$\frac{k_1}{\Gamma(z_1)\Gamma(z_2)\cdots\Gamma(z_n)}=\frac{k_j}{\Gamma(z_1)\cdots\widehat{\Gamma(z_j)}\cdots\Gamma(z_n)},$$

and the right hand side clearly does not depend on z_j. Hence $\frac{k_1}{\Gamma(z_1)\Gamma(z_2)\cdots\Gamma(z_n)}$ is constant and since $\frac{k_1}{\Gamma(z_1)\Gamma(z_2)\cdots\Gamma(z_n)}=\frac{g(z_1,\ldots,z_n)}{\Gamma(z_1)\cdots\Gamma(z_n)}$ which is constant then it follows that $f(z_1,\ldots,z_n)=c\cdot\frac{\Gamma(z_1)\cdots\Gamma(z_n)}{\Gamma(z_1+\ldots+z_n)}=cB(z_1,\ldots,z_n)$ and also note that $c=\frac{f(1,\ldots,1)}{B(1,\ldots,1)}=\frac{f(1,\ldots,1)}{(n-1)!}$.

□

3 Product formula

Using this we can give a easy proof for the following identity,

$$B(z_1,z_2)=\frac{1}{z_1+z_2-1}\prod_{k=1}^{\infty}\frac{k(z_1+z_2+k-2)}{(z_1+k-1)(z_2+k-1)}.$$

Clearly the function

$$f(z_1,z_2)=\frac{1}{z_1+z_2-1}\prod_{k=1}^{\infty}\frac{k(z_1+z_2+k-2)}{(z_1+k-1)(z_2+k-1)}$$

is analytic on $\{(z_1,z_2)\in\mathbb{C}^2\mid \text{Re }z_1,\ \text{Re }z_2>0\}$. and if we fix z_2 then since the product converges absolutely it follows that the continuous function $f(z,z_2)$ will be bounded on $\{z_1\in\mathbb{C}\mid \text{Re }z_1\}\times\{z_2\}$. Furthermore we can verify that